Hardy's Test of Local Realism

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Peter E. Wills

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John Essick

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work hard and learn, and I know she would have loved to see me at a school like Reed. I can only hope that one day I am as good a parent to my children as she was to me and my sister.

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Abstract

We perform a test of local realism using polarization-entangled photons from a spontaneous parametric down-conversion source. This test was designed by Mark Beck of Whitman College, and is based on the theoretical work of Lucien Hardy. We find local realism to be violated by more than 60 uncertainties. This experiment is relatively inexpensive, and can be performed by interested undergraduates.

Introduction

> *Cette harmonie que l'intelligence humaine croit découvrir dans la nature, existe-t-elle en dehors de cette intelligence? Non, sans doute, une réalité complètement indépendante de l'esprit qui la conçoit, la voit ou la sent, cest une impossibilité.*
>
> Does the harmony the human intelligence thinks it discovers in nature exist outside of this intelligence? No, beyond doubt, a reality completely independent of the mind which conceives it, sees or feels it, is an impossibility.
>
> Henri Poincaré[1]

> There is no quantum world. There is only an abstract physical description. It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature.
>
> Niels Bohr[2]

As quantum mechanics came of age in the early 20$^{\text{th}}$ century, scientists struggled to understand what lay behind their new theory of the microscopic world. The strikingly non-classical features of the new physical theory challenged the intuition of many, but the theory's unambiguous agreement with experiment solidified its position against attempts at philosophical debasement. The new era of quantum physics had begun.

For many decades, certain epistemological questions about quantum physics were not testable, and thus not considered within the scope of physics to answer. However, in 1964, a young Irish physicist named John Bell devised an approach which would, in principle, unambiguously test the empirical validity of local realism, a tenet of the classical conception of the physical world. Although it would be almost two decades before this test was experimentally realized, the results were found to be in clear violation of local realism. The once-philosophical debate had now been settled.

Twenty five years after the first experimental test of local realism, we are now able to carry out such tests in a modest undergraduate laboratory. Although we break no new ground in doing so, experimentally verifying this striking feature of quantum theory is an invaluable tool in coming to understand this theory of the microscopic world.

[1] Poincaré [1958].

[2] Quoted in Baggott [1992].

Chapter 1

Historical Background

1.1 The Meaning of Physical Theory

What is a physical theory? One possible response is that a physical theory is a quantitative description of the natural world, formulated in the language of mathematics. For Newtonian physics, it is fairly straightforward to assign a readily understandable physical meaning to most of the mathematical objects involved in the theory,[1] but what do we say when our mathematical apparatus seems to go beyond the reach our physical understanding? What does our physical theory "mean" in this case? Physicists and philosophers alike wrestled with these questions.[2]

Logical Positivism and the Vienna Circle

In considering the function of physical theory, a natural starting point is the work of the Vienna Circle,[3] a group of philosophers who gathered in Vienna in the early 1920s and would exert far-reaching influence on the philosophy of science. This group worked to give a clear definition of what was and was not scientific, in order to separate the physical from the metaphysical. They claimed that the only true knowledge is scientific knowledge, which is knowledge formulated in logical statements which are empirically verifiable (Achinstein and Barker [1969]).

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[1] A notable exception is the concept of energy. I think Feynman said it best:

> There is a fact, or if you wish, a law, governing natural phenomena that are known to date. There is no known exception to this law; it is exact, so far we know. The law is called conservation of energy; it states that there is a certain quantity, which we call energy, that does not change in manifold changes which nature undergoes. That is a most abstract idea, because it is a mathematical principle; it says that there is a numerical quantity, which does not change when something happens. It is not a description of a mechanism, or anything concrete; it is just a strange fact that we can calculate some number, and when we finish watching nature go through her tricks and calculate the number again, it is the same.

from Feynman [1964].

[2] And of course, the two vocations are far from mutually exclusive.

[3] The Vienna Circle included such notable members as Moritz Schlick, Rudolf Carnap, and Otto Neurath, among others.

Let us consider an example. The statement "When a tree falls in the woods, and no one is there to hear it, it does indeed make a sound" would *not* be considered a scientific statement by the logical positivists, as there is no conceivable way to verify it;[4] the statement is about the unobservable world, and therefore does not constitute knowledge. Similarly, a statement such as "this universe is just one of many universes" would not be scientific unless one could find a way to verify it through observation of our own universe. Through this criterion, the positivists sought to categorize the metaphysical as "not only false, but empty and cognitively meaningless"(Zalta [2010]).[5]

If we accept the doctrines of logical positivism, then it does not further our understanding to search for meaning in a physical theory beyond its predictions about experiments. If the theory conforms to experimental verification, then it is (for the time being) a correct theory.[6] Of course, the logical positivists did not put an end to the discussion; for many scientists, it was essential that they have a conviction regarding the interpretation of the physical theory they are considering. The issue became especially salient as quantum theory came of age in the late 1920s, as many of the mathematical objects used in the quantum theory were lacking a readily apparent physical analog. Ultimately, it would be up to physicists to construct an interpretation of the mathematical machinery that underlaid quantum theory.

1.2 Interpretations of Quantum Mechanics

One of the first major steps taken towards a physical theory of quanta was Einstein's explanation of the photoelectric effect via quantization of the electromagnetic field.[7][8] This was the first suggestion that an object could exhibit both wave and particle properties.[9] Brilliant minds are ever prone to abstraction, and soon this duality was being applied objects classically thought of strictly as particles, as de Broglie and Schrödinger began to consider the wave mechanics of particles.

[4] We will assume that recording the sound or employing some other remote listening mechanism is equivalent to someone being there to hear it.

[5] This criterion was not without its critics. Karl Popper urged a shift from a criterion of verifiability to one of *falsifiability*. Consider the statement "All electrons have equal charge." In order to actually verify this statement, one would have to observe every electron in the universe, which is not possible; however, it is clearly something we want to include as scientific knowledge. Notice that it is easily falsifiable; one would only need observe one electron with a charge not equal to all others.

[6] Of course, when considering two theories which both fully conform to experiment, we choose the more simple and elegant of the two; this is, however, an aesthetic choice, as both theories constitute true knowledge.

[7] This section adapts material primarily from Baggott [1992].

[8] The photoelectric effect is the ejection of electrons by a metal surface when light is shone on it. What puzzled many is that the energy of the ejected electrons depends on the frequency of the light incident on the surface, but not on the intensity. This could not be explained by a wave model of light.

[9] More accurately, light could act as *either* a wave or a particle, depending on the circumstances.

Schrödinger, Born, and the First Challenge to Realism

In formulating the dynamics of quantum mechanics, Schrödinger employed a mathematical object called a "wave function," which is governed by the famous Schrödinger equation. The question immediately arose: to what element of reality does this wave function correspond? Schrödinger's first idea was to equate the wave function with oscillation in the electromagnetic field, which would at large scales appear to our observational devices as a particle.[10] This approach immediately met with problems, as pointed out by Hendrik Lorentz.[11] Although Schrödinger could not defend such a literal view of the wave function, he was certainly not ready to accept the alternatives which would present themselves.

In 1926, Max Born suggested an alternative view of Schrödinger's wave function. Born's idea was that the wavefunction (or, more precisely, the square of the magnitude of the wave function) is a probability distribution, describing the probability of finding the particle within a certain interval. This idea was radical, for it made the wave function, rather than a physical quantity associated with the particle, be a representation of our *knowledge* of the particle. This resolved many of the issues that had arisen from Schrödinger's literal interpretation of the wave function, but it directly challenged determinism, which disturbed many thinkers of the time. Born claimed to have drawn inspiration from Einstein, but a 1926 letter from Einstein to Born betrayed Einstein's distaste for the new theory:

> Quantum mechanics is very impressive. But an inner voice tells me that it is not yet the real thing. The theory produces a good deal but hardly brings us closer to the secret of the Old One. I am at all events convinced that *He* does not play dice.[12]

Niels Bohr and the Copenhagen Interpretation

The next big step towards the modern understanding of quantum mechanics would come from a young Danish physicist named Niels Bohr. Bohr became the founder and first director of the Institute of Theoretical Physics at the University of Copenhagen in 1921, which would become a central institution in the advancement of theoretical physics. In 1922, he was awarded the Nobel Prize for his modeling of the atom.[13] In October 1926, Bohr invited Schrödinger and Heisenberg to join him in Copenhagen to discuss and debate issues regarding quantum mechanics. Schrödinger declined, but

[10] An analogy is readily drawn between this relation and the relation between ray optics and wave optics; ray optics is a functional way of analyzing optical systems with characteristic length scale much larger than the wavelength of light.

[11] When confined to a small region of space, the wave function (governed by the Schrödinger equation) is expected to spread into a uniform distribution. This is obviously not what happens to the **E** field of a confined electron. Furthermore, the wave function exists in *configuration space*, so that a composite wave function of a 2-particle system is a 6-dimensional object, whereas the **E** field of a two-particle composite state is obviously a 3-dimensional object.

[12] Einstein and Born [1971]

[13] His model, commonly known as the Bohr Model, theoretically explained the experimentally known Rydberg formula for the hydrogen emission spectrum.

Heisenberg joined Bohr; they were joined by Wolfgang Pauli, and the three together formulated an interpretation of quantum mechanics now known as the Copenhagen interpretation.[14]

The key principles of the Copenhagen interpretation are indeterminism, Born's statistical interpretation of the wave function, complementarity, and Bohr's correspondence principle (Zalta [2010]). Let us consider complementarity, for it is central to the Copenhagen group's idea.

Complementarity requires us to carefully examine how it is that we gain knowledge about the world; in that sense, it is similar to logical positivism. The idea behind complementarity is that an object can have two seemingly conflicting sets of properties, so long as those properties are observable only in mutually exclusive contexts. The most obvious example of this is wave-particle complementarity as seen with light. Light has wave-like properties when observed in a certain way (a double-slit experiment, for example) and has particle-like properties when observed in a different way (such as when observing the photoelectric effect). Assuming that the light has well-defined properties previous to measurement, how can we reconcile these sets of observations? Does the photon somehow rapidly switch between wave and particle properties in response to measurement? Bohr, in a decidedly positivist manner, would dismiss this question as ill-defined. Complementarity is essentially tied to the idea that is impossible for us to know things about the world beyond of our ability to observe it.

The idea of complementarity justified, in some sense, Heisenberg's famous uncertainty relation. The uncertainty relation was now a quantitative statement of the limits imposed on the measurement of complementary quantities.[15] To embrace complementarity was to embrace an element of indeterminism in the quantum world (although as the opening quote from Bohr indicates, it is difficult to say such a "world" even exists when it is entirely indeterminate previous to observation).

The final element of the Copenhagen interpretation is the correspondence principle, which requires that quantum mechanical laws reduce to classical laws when we consider the limit of large quantum numbers. This requirement assures that quantum physics and classical physics will not contradict each other when we are working with classical length-scales.

The Copenhagen interpretation was not without its critics. It essentially places a limit on what we are capable of knowing scientifically, which is a hard thing for any inquiring mind to accept. Einstein in particular, in his quest to uncover the "secret of the Old One," was certainly not willing to accept such an interpretation, especially

[14]Note that these physicists did not call their interpretation by this name, and did not even necessarily agree on all aspect of the interpretation. By the modern era, however, the refined version of this interpretation has become so deeply entrenched that it is often simply referred to as the "orthodox interpretation."

[15]The uncertainty relation can be viewed as a mathematical statement about the relation between the wavefunction of a particle in position space and the wavefunciton of the same particle in momentum space; more generally, there also exists an uncertainty relation between any localized function and its Fourier transform.

with its inherent threats to causality.[16] This disagreement between Bohr and Einstein sparked a series of debates over the meaning of quantum theory.

The Bohr-Einstein Debates

On October 24$^{\text{th}}$, 1927, Einstein, Bohr, and many other eminent physicists assembled in Brussels for the fifth Solvay conference. Although Einstein made no contributions to the formal proceedings, his intellectual sparring with Bohr served to lay the foundations for one of Einstein's strongest challenges to the Copenhagen interpretation of quantum theory.[17] Einstein proposed thought experiments (often referred to by the German term *Gedankenexperiment*) which challenged Bohr's complementarity and indeterminism, and Bohr came up with responses which held up the requirements of uncertainty, even using Einstein's own theory of general relativity against him![18] I will not describe the debates in detail; I refer the reader to Baggott [1992] §3.3 for a very readable account.

By the end of these debates, it appeared that Bohr had defended his ideas against the ingenious challenges brought against indeterminism by Einstein. However, the greatest challenge was yet to come. After conceding that quantum mechanics may not be inconsistent, Einstein set out to show that it was *incomplete*.

1.3 The Paradox of Einstein, Podolsky, and Rosen

In May of 1935, Einstein, along with Boris Podolsky and Nathan Rosen, published a paper titled 'Can quantum-mechanical description of reality be considered complete?' (A. Einstein [1935]). In this paper Einstein claimed to show that the wave function did not offer a complete description of reality. He explained the notion of reality at hand by saying "A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system." Einstein goes on to say that consideration of measurements performed on a certain system possessing two complementary observables leads to the result that if the wave function *is* in fact a complete description of the system, then there *must* exist two complementary quantities which have simultaneous reality! This result is reached through careful design of the system and selection of the two complementary quantities to which we attribute reality.

Consider a system of two particles which starts out in a composite state of zero spin angular momentum.[19] This is not difficult to conceive of; one such state would be a hydrogen molecule with its electrons spin-paired in the ground state. Suppose that, via some interaction, the composite system dissolves, and the two particles separate.

[16] The Copenhagen school would dismiss causality as a classical requirement, and a logical positivist would similarly dismiss it as a metaphysical requirement.

[17] I refer, of course, to the famous paradox of Einstein, Podolsky, and Rosen.

[18] This refers to Bohr's response to Einstein's 'photon box' thought experiment.

[19] Einstein's original argument involves the fuzzy quantities of momentum and position measurements; we will present here David Bohm's simplified version of the paradox, which uses spin-$\frac{1}{2}$ particles.

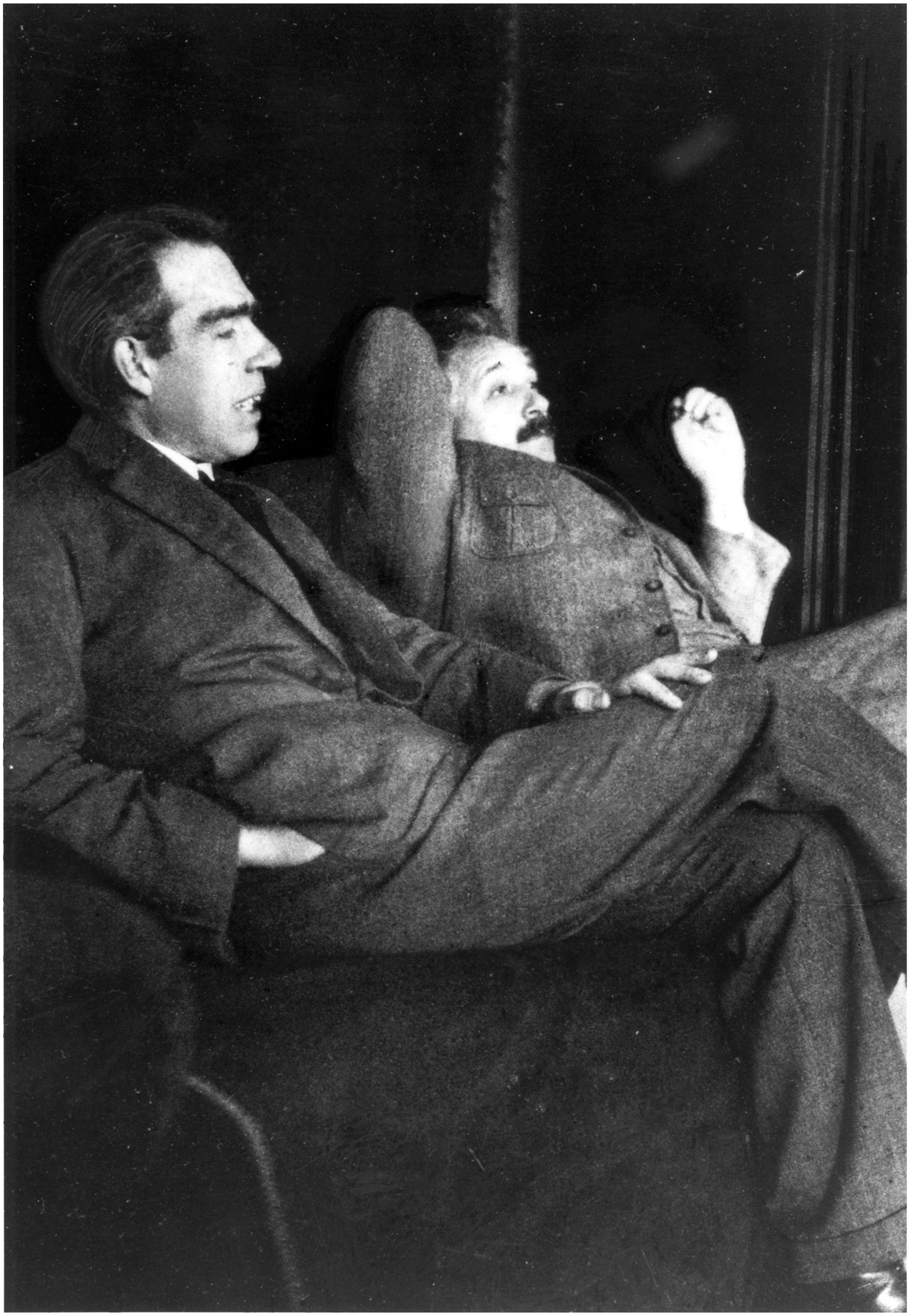

Figure 1.1: Bohr and Einstein in the home of Ehrenfest, December 1925

Note that by conservation of angular momentum, the spins of the two particles are now anti-correlated.[20] If we measure the spin of particle a along some axis $\hat{\mathbf{n}}$ (which can only take on two values, $\pm\frac{1}{2}\hbar$), we know with certainty the value of the spin of b along $\hat{\mathbf{n}}$. We thus must say that, according to Einstein's definition, the spin of b along $\hat{\mathbf{n}}$ is an element of physical reality. But since $\hat{\mathbf{n}}$ is arbitrary, we must attribute physical reality to the spin of b along every axis![21] However, according to quantum mechanics, the spin components of a particle measured along two perpendicular axes are complementary, and cannot be known with exact precision simultaneously. Thus, the quantum mechanical description is incomplete, for it does not allow us simultaneous knowledge of two values which have physical reality (as defined by Einstein).[22]

This struck at the core of Bohr's idea of complementarity. If it was to be saved, he would have to invoke some kind of action at a distance between the two particles, where measurement of the spin of particle a would have the same effect upon particle b as if we were to measure the spin of particle b in the first place! Bohr seemed to find a way around this, and countered with an article of the same name as that of Einstein, Podolsky, and Rosen, which challenged their definition of physical reality on the basis of complementarity. The debate continued, however; was Bohr's reply valid, or had Einstein shown the incompleteness of quantum mechanics? The community did not even think that a definitive answer was possible until decades later, when Bell proposed a quantitative way of comparing the two views.

1.4 Hidden Variables and Bell's Theorem

The work of John S. Bell in his paper titled 'On the Einstein Podolsky Rosen Paradox' (Bell [1964]) did not make a big splash when it was first published in 1964. Its importance was only slowly recognized, but from the 1970s to today it has been considered absolutely essential to any study of foundational quantum mechanics. The idea behind Bell's inequality will here be presented in a very qualitative manner; a rigorous derivation would not serve to clarify the matter, as we do not use this specific inequality in the experiment at hand. However, the curious reader is referred to Appendix B for one derivation.

Hidden Variable Theories

Bell's theorem compares quantum mechanics to a class of theories known as "hidden variable theories." Suppose that quantum mechanics is, in fact, an incomplete theory,

[20] Correlation generally means that we can infer the value of one quantity from the value of the other. We use it here in a more specific way: two quantities , a and b, are correlated if $a = b$ always holds; they are anti-correlated if $a = -b$ always holds.

[21] Note that we don't have to measure the spin of a along $\hat{\mathbf{n}}$ (and thus infer the spin of b along $\hat{\mathbf{n}}$) to establish the reality of the spin of b along $\hat{\mathbf{n}}$; it is enough that we *could* predict it with certainty without disturbing b.

[22] In other words, a complete theory would describe precisely all - and specifically, these two - elements of physical reality; quantum mechanics is only capable of describing them up to a finite uncertainty.

as was thought by Einstein, Podolsky, and Rosen. In that case, there must be some parameter or set of parameters inaccessible to us, which fully describes the physical state of the system.[23] This is to say that before measurement, an observable *does* have a certain value - we are simply ignorant of it until measurement. Quantum mechanics then describes probabilities, but it doesn't tell the whole story; the values that quantum theory describes as probabilistic are, in fact, well determined, and we no longer must think of God playing dice.

These kinds of theories are not unique to quantum mechanics. Consider the example of Boltzmann's statistical mechanics, which applied probability theory to study large thermodynamic systems. In this theory we only speak of the system as a whole, rather than analyze each individual constituent. We thus know from statistical mechanics what the *probability* is of any one particle in the system having a certain position or velocity, but we do not know *with certainty* what the dynamics of any one particle will be. In this case, then, the hidden variable would specify the dynamics of each particle.[24]

With this rough definition of a hidden variable theory in mind, we are now prepared to understand the momentous result of Bell.

Bell's Theorem

Loosely put, Bell's theorem states that no local realistic theory can reproduce all the results of quantum mechanics. Why is this remarkable? What Einstein was attempting to show with his paradox was that quantum mechanics must include some set of hidden parameters. In defending his position, Bohr was defending the possibility of complementarity and the Copenhagen interpretation, but he was not saying that a hidden variable theory was impossible.[25] What is radical about Bell's theorem is that it doesn't simply allow us to leave or take the hidden variable concept; it states a hidden variable theory will produce manifestly different results from quantum mechanics taken without hidden variables! With this paper Bell forced the issue of hidden variables into the realm of the empirical; for the first time, it was considered possible to falsify local realism.

Bell's inequality dealt with two spin-entangled particles as described above with the EPR paradox. This arrangement, although ingenious in theory, would prove prohibitively difficult to realize in practice. It was not until 1982 that a paper was published which rigorously tested Bell's inequality in an experimental laboratory.

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[23] This is to say that this parameter would tell us *exactly* what quantum mechanics as it stands can only tell us *probabilistically*.

[24] It is interesting that a notable positivist, Ernst Mach, opposed thinking of individual atoms or molecules; he insisted that we only think of the thermodynamic system which they constitute. However, the advancement of experimental technology allowed us to probe the individual entities of a thermodynamic system, making Mach's views untenable (Baggott [1992]).

[25] From a positivist's point of view, however, if the hidden variable is not necessary, it is metaphysical and does not belong within a physical theory.

1.5 Experimental Verification

Bell pushed us closer to an empirical falsification of all hidden variable theories. He did not take us all the way, however; the experimental techniques needed to perform this experiment accurately were still years away. What experimenters needed was a source that would output two particles in high volume, in a near-ideal singlet state. One such source turned out to be the atomic cascade of the calcium atom.

In the ground state of the calcium atom, the outermost orbital is filled with two spin-paired electrons. The net spin angular momentum of the two electrons is zero. If we use a photon to excite one of the electrons, the photon imparts a quanta of angular momentum onto the system, which appears as orbital angular momentum (since we cannot change the spin angular momentum of the electron). Now suppose we excite both electrons. Although there are multiple configurations of angular momentum which could come of this double excitation, we are interested in the state where the net angular momentum is zero. The doubly-excited calcium atom will undergo a rapid return to the ground state, emitting two photons in the process. This rapid double emission is referred to as a 'cascade.' Since the net angular momentum before the cascade is zero, we know that the two emitted photons must have opposite circular polarization.

In 1982, Alain Aspect of the University of Paris used such an atomic cascade to test Bell's inequality and show that nature did not adhere to the predictions of a hidden variable theory (Aspect et al. [1982]). I will not here include a detailed discussion of the experiment; for a more readable account than the original paper, the reader is referred to Baggott [1992] §4.4. What is important is that, from their measurements, Aspect and his colleagues violated the predictions of local realism by over 46 standard deviations.[26] Hidden variables theories has been decisively disproven in a purely empirical fashion.

1.6 Bell's Inequality Experiments Today

Did all work in the field come to a stop with the publication of Aspect et al. [1982]? Of course not, for there were still small modifications to be made, and many were interested in making the experimental apparatus smaller and more elegant, so that these results could be more easily reproduced. Additionally, the Aspect experiment tested specifically the version of Bell's inequality found in Clauser et al. [1970]. Physicists continued to derive many different versions of Bell's inequality, and experimenters were constantly challenged to find ways to realize these inequalities experimentally.[27] This thesis is concerned with one such test of local realism, as derived in Hardy [1993], and a corresponding experimental realization, envisioned by Mark Beck and his colleagues in Carlson et al. [2006].

[26] The form of Bell's inequality used by Aspect was the generalization first derived by Clauser et al. [1970].

[27] When I say something is a "version of Bell's inequality," I only mean that it is a quantitative test of local realism.

Chapter 2

Theory

We now derive a quantum state which displays a testable violation of local realism. The incompatibility of the state with assumptions of local realism are shown, and the conditions for maximal violation of local realism are derived. The following is adapted from Hardy [1993].[1]

2.1 Derivation of the Hardy State

Suppose we have a 2-photon polarization-entangled state, described in some basis by[2]

$$|\Psi\rangle = a\ |H\rangle_1 |H\rangle_2 + e^{i\phi} b\ |V\rangle_1 |V\rangle_2 , \tag{2.1}$$

where $|H\rangle_i$ ($|V\rangle_i$) denotes that the i^{th} element of the system is horizontally (vertically) polarized, a and b are two real numbers, and the complex exponential is a unit-amplitude phase factor. Normalization then gives

$$a^2 + b^2 = 1. \tag{2.2}$$

In our experiment, we set the phase so that $\phi = 0$, and thus

$$|\Psi\rangle = a\,|H\rangle_1 |H\rangle_2 + b\,|V\rangle_1 |V\rangle_2 \tag{2.3}$$

We now consider a new polarization basis, denoted $\widehat{H}$ and $\widehat{V}$, which for particle 1 is rotated an angle θ_1 from the original basis. For particle 2, we rotate the basis by

[1]As originally written by Hardy, this account is quite opaque. Without clear motivation, he moves through pages of mathematics, arriving finally at a particular expression of the two-particle entangled state; he then makes clear how this state violates local realism. This ends-justify-the-means approach is certainly not invalid, and perhaps is the cleanest way to go about such an argument; however, it is certainly not the most illuminating. Although I take roughly the same approach in this chapter, I might advise the befuddled reader to skip ahead to §2.2, and come to see how the Hardy state violates local realism, before reading §2.1, and seeing how we arrive at such a state.

[2]Hardy worked out the state relative to general quantum mechanical observables; we work specifically with polarization, as it is the observable relevant to our experimental implementation.

θ_1 in the opposite direction.[3] Our new polarization basis vectors are then related to the original basis vectors by

$$|\widehat{H}\rangle_i = \cos\theta_1 \; |H\rangle_i \pm \sin\theta_1 \; |V\rangle_i \,, \tag{2.4}$$
$$|\widehat{V}\rangle_i = \mp\sin\theta_1 \; |H\rangle_i + \cos\theta_1 \; |V\rangle_i \,, \tag{2.5}$$

where the top signs correspond to particle 1 and the bottom signs correspond to particle 2. There are then inverse relations

$$|H\rangle_i = \cos\theta_1 \, |\widehat{H}\rangle_i \mp \sin\theta_1 \, |\widehat{V}\rangle_i, \tag{2.6}$$
$$|V\rangle_i = \sin\theta_1 \; \pm |\widehat{H}\rangle_i + \cos\theta_1 \, |\widehat{V}\rangle_i. \tag{2.7}$$

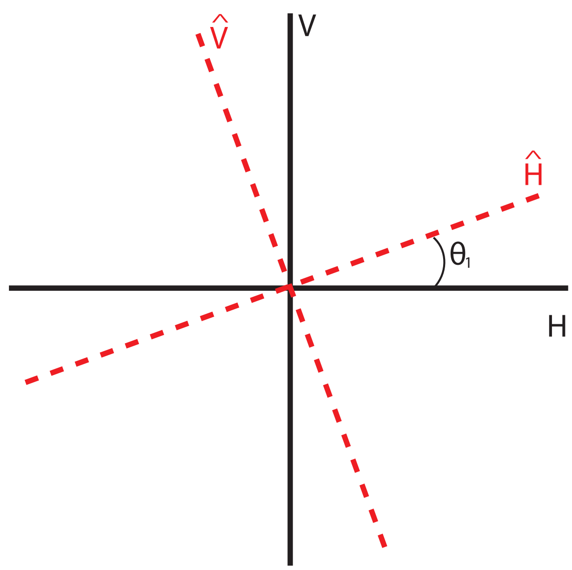

Figure 2.1: Our basis $\widehat{H}$, $\widehat{V}$ is rotated by an angle θ_1 relative to our original H, V basis.

Substituting these inverse relations into Equation (2.3), we have

$$\begin{aligned} |\Psi\rangle = \left(a\cos^2\theta_1 - b\sin^2\theta_1\right) \; |\widehat{H}\rangle_1|\widehat{H}\rangle_2 + \left(b\cos^2\theta_1 - a\sin^2\theta_1\right) \; |\widehat{V}\rangle_1|\widehat{V}\rangle_2 + \\ (a\sin\theta_1\cos\theta_1 + b\sin\theta_1\cos\theta_1) \; \left(|\widehat{H}\rangle_1|\widehat{V}\rangle_2 - |\widehat{V}\rangle_1|\widehat{H}\rangle_2\right) \end{aligned} \tag{2.8}$$

For reasons that will become clear later, the Hardy state requires that the first coefficient vanishes, i.e.

$$a\cos^2\theta_1 - b\sin^2\theta_1 = 0,$$

or

$$\frac{\sin^2\theta_1}{a} = \frac{\cos^2\theta_1}{b} = k^2,$$

[3]If this confuses you, hang on - it is a necessary complication, if our state $|\Psi\rangle$ is to exhibit the properties we desire. I might recommend the confused reader to §2.2 to examine these properties.

where k is some real constant.[4] We can equivalently say that

$$\sin\theta_1 = k\sqrt{a} \qquad \text{and} \qquad \cos\theta_1 = k\sqrt{b}. \tag{2.9}$$

These relations can be used in conjunction with the fact that $\sin^2\theta_1 + \cos^2\theta_1 = 1$ to show that

$$k^2 = \frac{1}{a+b}. \tag{2.10}$$

Substituting Equations (2.9) and (2.10) in (2.3), we can now express our state as

$$|\Psi\rangle = (b-a)\,|\widehat{V}\rangle_1|\widehat{V}\rangle_2 + \sqrt{ab}\,\left(|\widehat{H}\rangle_1|\widehat{V}\rangle_2 - |\widehat{V}\rangle_1|\widehat{H}\rangle_2\right), \tag{2.11}$$

which can be written as (dropping a phase factor of -1)

$$|\Psi\rangle = \left(\frac{\sqrt{ab}}{\sqrt{a-b}}|\widehat{H}\rangle_1 - \sqrt{a-b}\,|\widehat{V}\rangle_1\right)\times\left(\frac{\sqrt{ab}}{\sqrt{a-b}}\,|\widehat{H}\rangle_2 + \sqrt{a-b}\,|\widehat{V}\rangle_2\right) - \frac{ab}{a-b}|\widehat{H}\rangle_1|\widehat{H}\rangle_2. \tag{2.12}$$

We now consider another change of basis, rotating by an angle θ_2 relative to our θ_1 basis for each particle. We will denote horizontal and vertical polarization in this basis by $|\widetilde{H}\rangle$ and $|\widetilde{V}\rangle$. The angle θ_2 is given by

$$\cos\theta_2 = \frac{\sqrt{ab}}{\sqrt{1-ab}} \qquad \text{and} \qquad \sin\theta_2 = \frac{a-b}{\sqrt{1-ab}} \tag{2.13}$$

Our new basis vectors are thus

$$|\widetilde{H}\rangle_i = \cos\theta_2\,|\widehat{H}\rangle_i \mp \sin\theta_2\,|\widehat{V}\rangle_i, \tag{2.14}$$

$$|\widetilde{V}\rangle_i = \pm\sin\theta_2\,|\widehat{H}\rangle_i + \cos\theta_2\,|\widehat{V}\rangle_i, \tag{2.15}$$

with inverse relations

$$|\widehat{H}\rangle_i = \cos\theta_2\,|\widetilde{H}\rangle_i \pm \sin\theta_2\,|\widetilde{V}\rangle_i, \tag{2.16}$$

$$|\widehat{V}\rangle_i = \mp\sin\theta_2\,|\widetilde{H}\rangle_i + \cos\theta_2\,|\widetilde{V}\rangle_i. \tag{2.17}$$

Again, the top sign denotes the transformation for particle 1, and the bottom sign corresponds to particle 2. Using these definitions of the θ_2 bases and Equation (2.12), we express our state as

$$|\Psi\rangle = N\left(|\widetilde{H}\rangle_1|\widetilde{H}\rangle_2 - \cos^2\theta_2|\widehat{H}\rangle_1|\widehat{H}\rangle_2\right). \tag{2.18}$$

where N is defined as

$$N = \frac{1-ab}{a-b}. \tag{2.19}$$

[4]This is why we require that the bases rotate in opposite directions for the two particles - otherwise we would be unable to have $a\cos^2\theta_1 - b\sin^2\theta_1 = 0$.

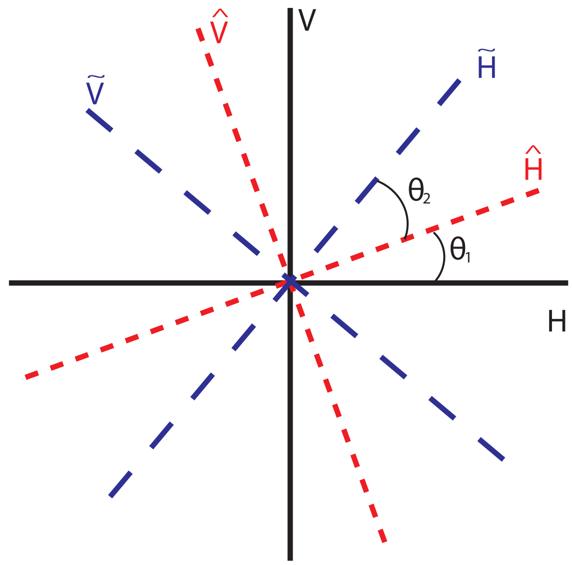

Figure 2.2: Our basis $\widetilde{H}$, $\widetilde{V}$ is rotated by an angle θ_2 relative to our original $\widehat{H}$, $\widehat{V}$ basis.

Using the basis transformation rules given in Equations (2.14) through (2.17) and the form of $|\Psi\rangle$ given in Equation (2.18), we can express of our state $|\Psi\rangle$ in four equivalent forms:

$$|\Psi\rangle = N\left[(\sin\theta_2\cos\theta_2)\left(|\widehat{V}\rangle_1|\widehat{H}\rangle_2 - |\widehat{H}\rangle_1|\widehat{V}\rangle_2\right) - \sin^2\theta_2\,|\widehat{V}\rangle_1|\widehat{V}\rangle_2\right], \tag{2.20}$$

$$|\Psi\rangle = N\left[|\widetilde{H}\rangle_1\left(\cos\theta_2\,|\widehat{H}\rangle_2 + \sin\theta_2\,|\widehat{V}\rangle_2\right) - \cos^2\theta_2\left(\cos\theta_2\,|\widetilde{H}\rangle_1 + \sin\theta_2\,|\widetilde{V}\rangle_1\right)\,|\widehat{H}\rangle_2\right], \tag{2.21}$$

$$|\Psi\rangle = N\left[\left(\cos\theta_2\,|\widehat{H}\rangle_1 - \sin\theta_2\,|\widehat{V}\rangle_1\right)|\widetilde{H}\rangle_2 - \cos^2\theta_2\,|\widehat{H}\rangle_1\left(\cos\theta_2\,|\widetilde{H}\rangle_2 - \sin\theta_2\,|\widetilde{V}\rangle_2\right)\right], \tag{2.22}$$

$$|\Psi\rangle = N\left[|\widetilde{H}\rangle_1|\widetilde{H}\rangle_2 - \cos^2\theta_2\left(\cos\theta_2\,|\widetilde{H}\rangle_1 + \sin\theta_2\,|\widetilde{V}\rangle_1\right)\left(\cos\theta_2\,|\widetilde{H}\rangle_2 - \sin\theta_2\,|\widetilde{V}\rangle_2\right)\right]. \tag{2.23}$$

2.2 Incompatibility with Local Realism

We now look to our operators corresponding to measurement of horizontal polarization in the θ_1 basis and vertical polarization in the θ_2 basis. Note that these operators function on a single particle; we will use pairs of such operators to observe the polarization of our 2-particle system, for example,

$$\widehat{H}_i = |\widehat{H}\rangle_i\langle\widehat{H}|_i, \tag{2.24}$$

$$\widetilde{V}_i = |\widetilde{V}\rangle_i\langle\widetilde{V}|_i. \tag{2.25}$$

These operators can take values of 0 or 1, representing non-detection or detection by an appropriate polarization detector.[5] We now will briefly digress, and discuss conventions of probability calculation and notation.

The probability amplitude that an outcome A will be measured for an observable $\hat{A}$ on a quantum system $|\Psi\rangle$ is given by $|\langle A\,|\Psi\rangle|^2$. We denote this probability $P(A)$:

$$P(A) = |\langle A\,|\Psi\rangle|^2 .$$

For a two particle system, we can observe joint probabilities, which is to say the probability that we will measure both A on particle 1 and B on particle 2. We denote this probability $P(A_1, B_2)$, where the subscripts denote the particle in the system upon which the operator acts. This probability is then given by

$$P(A_1, B_2) = |(\,_1\langle A|\,\,_2\langle B|)\,|\Psi\rangle|^2 .$$

Another probability we are interested in for a system of multiple particles is the conditional probability, which is to say the probability that we will measure A_1 *given that we measure* B_2. We denote this $P(A_1|B_2)$. This is given by simply dividing the joint probability by the probability of B_2:

$$P(A_1|B_2) = \frac{P(A_1, B_2)}{P(B_2)}. \tag{2.26}$$

We can now make four observations about the Hardy state, based on the four representations of $|\Psi\rangle$, as shown in Equations (2.20), (2.21), (2.22), and (2.23).

1. Looking to the representation of $|\Psi\rangle$ shown in Equation (2.20), we see that there is zero probability that we will measure horizontal polarization for both particles *in the* θ_1 *basis*:

$$\left|\left(\,_1\langle\widehat{H}|\,_2\langle\widehat{H}|\right)|\Psi\rangle\right|^2 = 0 \tag{2.27}$$

   We designed the system as such when we required that $a\cos^2\theta_1 - b\sin^2\theta_1 = 0$.

2. Looking to the representation of $|\Psi\rangle$ shown in Equation (2.21), we see that $P(\widetilde{V}_1) = P(\widehat{H}_2, \widetilde{V}_1)$, and so

$$P\left(\widehat{H}_2|\widetilde{V}_1\right) = 1 \tag{2.28}$$

   This is to say that if we measure vertical polarization on particle 1 *in the* θ_2 *basis*, then we are sure to measure horizontal polarization on particle 2 *in the* θ_1 *basis.*

[5]Note that we have here listed two of eight operators in this class; one could measure either horizontal or vertical polarization, in either of the two bases, on either particle.

3. Similarly, looking to the representation of $|\Psi\rangle$ shown in Equation (2.22), if we measure vertical polarization on particle 2 *in the θ_2 basis*, then we are sure to measure horizontal polarization on particle 1 *in the θ_1 basis*. It is easily seen that $P(\widetilde{V}_2) = P(\widehat{H}_1, \widetilde{V}_2)$, and so from Equation (2.26) we see that

$$P\left(\widehat{H}_1|\widetilde{V}_2\right) = 1 \tag{2.29}$$

4. Finally, looking to the representation of $|\Psi\rangle$ seen in Equation (2.23), the joint probability of measuring vertical polarization on both photons *in the θ_2 basis*,

$$\begin{aligned} P(\widetilde{V}_1, \widetilde{V}_2) &= \left| \left({}_1\langle \widetilde{V}|\, {}_2\langle \widetilde{V}| \right) |\Psi\rangle \right|^2 \\ &= \left| N \cos^2\theta_2 \sin^2\theta_2 \right|^2 \end{aligned} \tag{2.30}$$

We are now in a position to derive a contradiction using only the assumption of local realism. Assuming local realism is to assume that a measurement made on one particle in the system cannot affect the outcome of a measurement made on the other particle in the system; that is to say, each particle in the composite state has a well-defined polarization independent of the other. If this is indeed the case, then there is some parameter or set of parameters, which we will denote λ, which determines the polarization of each particle in the system. This is called the "hidden variable," for it is not accessible to us through application of quantum theory; we can only assert that such a parameter or set of parameters *must* exist in order for local realism to hold.

We will now discuss a specific instance of measurement. In this case, we can measure only values of 1 or 0.[6] Consider an event in which we measure both photons to have vertical polarization in the θ_2 basis. We know this is allowed, as the probability of this event is non-zero in Equation (2.30) for $a \neq b$ (see Equation 2.19). Since the hidden variable λ determines the result of measurement on the system, we write

$$\widetilde{V}_1(\lambda)\widetilde{V}_2(\lambda) = 1. \tag{2.32}$$

Now suppose that we had chosen to measure the polarization of particle 1 *in the θ_1 basis*. This measurement event *cannot* affect the outcome of our measurement on particle 2, based on our assumption of local realism. From Equation (2.29), we see that since we still would measure $\widetilde{V}_2(\lambda) = 1$, we necessarily have

$$\widehat{H}_1(\lambda)\widetilde{V}_2(\lambda) = 1. \tag{2.33}$$

Similarly, if we had decided to measure polarization in the θ_1 basis on particle 2, we would have

$$\widetilde{V}_1(\lambda)\widehat{H}_2(\lambda) = 1. \tag{2.34}$$

[6] We measure polarization by passing particles through polarizers with detectors behind them; thus, the values 1 or 0 correspond to detection events at the detectors behind the appropriate polarizers. As an example: $\widetilde{V}_1\widetilde{V}_2 = 1$ would indicate that vertical polarization in the θ_2 basis was measured on both particles. Similarly, $\widetilde{V}_1\widetilde{V}_2 = 0$ would indicate that at least one (if not both) of the particles was not detected at the detector behind the vertical polarizer.

Finally, suppose that we had chosen to measure both particles in the θ_1 basis. We have shown in Equations (2.33) and (2.34), for this measurement event, which corresponds to the hidden variable λ, that $\widehat{H}_{\theta_1 1}(\lambda) = 1$ and $\widehat{H}_{\theta_1 2}(\lambda) = 1$. Therefore, if we had chosen to measure both particles in the θ_1 basis, we would have measured

$$\widehat{H}_1(\lambda)\widehat{H}_2(\lambda) = 1, \tag{2.35}$$

which directly contradicts Equation (2.27).

This contradiction rests on the assumption of local realism, which is the assumption that we can choose to change the basis of our measurement "at the last minute."[7] We therefore conclude that local realism is incompatible with the predictions of quantum theory in this case.

2.3 The Maximal Hardy State

We have shown the incompatibility of the Hardy state with local realism by considering one specific event. But if we follow this logic on a larger scale, *any* $(\widetilde{V}_1, \widetilde{V}_2)$ detection *would* have been a $(\widehat{H}_1, \widehat{H}_2)$ detection, had we repositioned our detectors appropriately. Saying this in terms of probabilities, what we have just shown is that for our state $|\Psi\rangle$, local realism predicts

$$P(\widehat{H}_1, \widehat{H}_2) \geq P(\widetilde{V}_1, \widetilde{V}_2). \tag{2.36}$$

What we must do experimentally is measure these two probabilities and compare them.[8] The greatest violation of Equation (2.36) will occur when we maximize $P(\widetilde{V}_1, \widetilde{V}_2)$. From Equations (2.30), (2.19) and (2.13), we have

$$\begin{aligned} P(\widetilde{V}_1, \widetilde{V}_2) &= \left| N \cos^2\theta_2 \sin^2\theta_2 \right|^2 \\ &= \left[\frac{(a-b)\,ab}{1-ab} \right]^2 . \end{aligned}$$

We can express this in terms of a single variable $x = ab$:

$$\begin{aligned} \left(\frac{(a-b)\,ab}{1-ab} \right)^2 &= \frac{(ab)^2(a^2+b^2-2ab)}{(1-ab)^2} \\ &= \frac{x^2\,(1-2x)}{(1-x)^2}, \end{aligned}$$

which is maximized when

$$x = \frac{3-\sqrt{5}}{2}, \tag{2.37}$$

[7]This is to say we change the basis of measurement on (say) particle 1 such that no sub-luminal communication could alert particle 2 to this change before we measure the polarization on both particles.

[8]This discussion is taken in part from Carlson et al. [2006] as well as Hardy [1993]

yielding

$$P(\widetilde{V}_1, \widetilde{V}_2) = 0.09017 \approx 9\%. \tag{2.38}$$

We now have a set of two equations governing our maximal a and b:

$$a^2 + b^2 = 1 \tag{2.39}$$

$$2ab = 3 - \sqrt{5} \tag{2.40}$$

A solution to this set of equations is

$$a \approx 0.421 \qquad \text{and} \qquad b \approx 0.907 \tag{2.41}$$

The other three solutions consist in negation or reversal of our values for a and b. Of course, the states corresponding to these other solutions lead to equal violation of Equation (2.36). From Equations (2.2), (2.9) and (2.13), we can find the angles θ_1 and θ_2 corresponding to the above values for a and b:

$$\theta_1 \approx 34.27^\circ \qquad \text{and} \qquad \theta_2 \approx 38.17^\circ \tag{2.42}$$

Chapter 3

Experimental Considerations and Techniques

3.1 The Quantity H

How are we to measure the extent to which we violate the inequality shown in Equation (2.36)? We could simply compare two probabilities, which obey different inequalities for local realism and quantum theory:

$$P(\widehat{H}_1, \widehat{H}_2) \geq P(\widetilde{V}_1, \widetilde{V}_2) \qquad \text{(local realism)}$$
$$P(\widehat{H}_1, \widehat{H}_2) < P(\widetilde{V}_1, \widetilde{V}_2) \qquad \text{(QM)}$$

However, this does not ensure that we are measuring photons in the Hardy state. In order to be sure that we are measuring the Hardy state (or as close to the Hardy state as is possible), our test must quantitatively ensure that Observations 1-4 are satisfied. We can rewrite observations 2 and 3 in a form more amenable to measurement as follows:[1]

2. Measurement of vertical polarization on particle 2 *in the θ_2 basis* implies measurement of horizontal polarization on particle 1 *in the θ_1 basis.* We can equivalently express this by saying that we will never simultaneously measure vertical polarization on particle 1 *in the θ_1 basis* and vertical polarization on particle 2 *in the θ_2 basis*:
$$P(\widehat{V}_1, \widetilde{V}_2) = 0 \tag{3.1}$$

3. We can similarly express observation 3 by saying that we will never simultaneously measure vertical polarization on particle 1 *in the θ_2 basis* and vertical polarization on particle 2 *in the θ_1 basis*:
$$P(\widetilde{V}_1, \widehat{V}_2) = 0 \tag{3.2}$$

[1]Note that these two revisions are based on the theorem of logic

$$a \rightarrow \sim b \iff \sim(a \wedge b)$$

We now have four quantities which we can measure experimentally. We wish to maximize our measurement of the probability shown in Equation (2.30), while minimizing our measurement of the probabilities shown in Equations (2.27), (3.1), and (3.2). It is thus natural to define the quantity H:

$$H = P(\widetilde{V}_1, \widetilde{V}_2) - P(\widehat{H}_1, \widehat{H}_2) - P(\widetilde{V}_1, \widehat{V}_2) - P(\widehat{V}_1, \widetilde{V}_2) \tag{3.3}$$

Local realism is violated when $H \geq 0$.

3.1.1 Measurement Angles

Experimentally, it is convenient to describe these probabilities in terms of the angles which the wave plates will be set at during measurement. As discussed in Chapter 2, in order to satisfy $P(\widehat{H}_1, \widehat{H}_2) = 0$, the polarizers on the two particles must rotate in opposite directions.[2]

Our apparatus is designed to read $P(H_1, H_2)$. Therefore, if we wish to measure *horizontal* probability in a basis $(H_{\theta_j}, V_{\theta_j})$, we rotate our waveplates by an angle θ_j, and if we wish to measure *vertical* probability in a basis $(H_{\theta_j}, V_{\theta_j})$, we rotate our wave plates by an angle $\theta_j \pm 90°$.

With that in mind, we define the angle of $|H\rangle$ as $0°$. We then rewrite Equation (3.3) using $\widehat{H} \to \alpha$, $\widehat{V} \to \alpha_\perp$, $\widetilde{V} \to \beta$,[3] so

$$H = P(-\beta, \beta) - P(-\alpha, \alpha) - P(-\beta, \alpha_\perp) - P(-\alpha_\perp, \beta) \tag{3.4}$$

What are the angles α and β? Our apparatus is set up so that we measure $P(H_1, H_2)$ with the half wave plates set to zero. Comparing Equations (3.3) and (3.4), $\widehat{H} \to \alpha$, so $\alpha = \theta_1$. As for β, we know that the $(\widetilde{H}, \widetilde{V})$ basis is rotated from our original basis by an angle of $(\theta_1 + \theta_2)$. Again, comparing Equations (3.3) and (3.4), $\widetilde{V} \to \beta$, so $\beta = \theta_1 + \theta_2 \pm 90°$. We choose the negative sign to get

$$\alpha = 34.27° \qquad \text{and} \qquad \beta = -17.56° \tag{3.5}$$

For convenience, we absorb the negative sign on β into our expression for H, so

$$H = P(\beta, -\beta) - P(-\alpha, \alpha) - P(\beta, \alpha_\perp) - P(-\alpha_\perp, -\beta), \tag{3.6}$$

and

$$\alpha = 34.27° \qquad \text{and} \qquad \beta = 17.56°. \tag{3.7}$$

[2]See §2.1

[3]We set the particle 1 axes to rotate through a negative angle and particle 2 axes to rotate through a positive angle so that Equation 3.6 is consistent with Carlson et al. [2006]. Of course, we could just as well have rotated the particle 1 axes a positive angle and the particle 2 axes a negative angle.

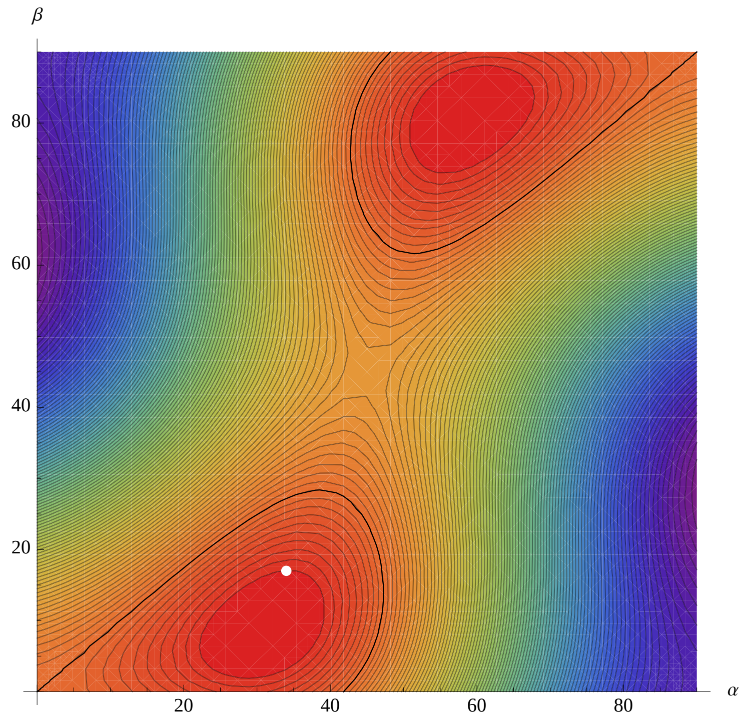

Figure 3.1: Plot of H vs. α, β. The white point is the (α, β) value used in our experiment, and the thick contour is $H = 0$.

A plot of H is shown in Figure 3.1. This plot is generated using[4]

$$P(\alpha, \beta) = (a \cos\alpha \cos\beta + b \sin\alpha \sin\beta)^2, \tag{3.8}$$

to calculate all four probabilities involved in Equation (3.6). Our point of measurement is marked, along with the $H = 0$ contour. One can see from the diagram that we do not measure at the precise maximum of H, which occurs around $(\alpha, \beta) = (31°, 9°)$.

Why do these angles not agree with those calculated in Chapter 2? Figure 3.1 is generated by allowing all the probabilities to vary freely; there is no constraint that $P(-\alpha, \alpha) = P(\beta, \alpha_\perp) = P(-\alpha_\perp, -\beta) = 0$, as there was in Chapter 2. By allowing these probabilities to be non-zero, we can increase the value of $P(\beta, -\beta)$ significantly and thus increase H. However, in practice, we find that, compared to measurement at $(\alpha, \beta) = (34°, 17°)$, measurement at $(\alpha, \beta) = (31°, 9°)$ increases the probabilities we wish to *minimize* significantly, while not having much affect on the probability we wish to *maximize*. The crux of this experiment is to get the three probabilities $P(-\alpha, \alpha)$, $P(\beta, \alpha_\perp)$, and $P(-\alpha_\perp, -\beta)$ as low as possible; in practice, it is this that gives us maximal violation of Equation (3.6).

[4]See Appendix A of Carlson et al. [2006] for a derivation.

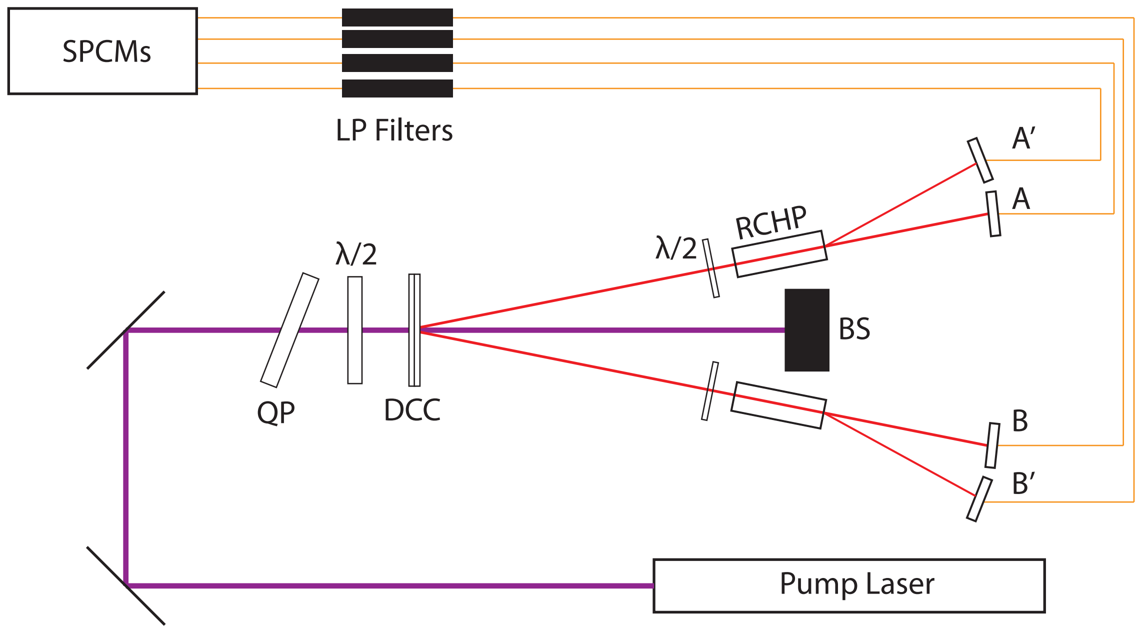

Figure 3.2: Experimental apparatus for determining H. Here QP is the quartz plate, DCC is the downconversion crystal, SPCM is the single photon counting module, RCHP is the Rochon polarizer, and $\lambda/2$ is the half-wave plate.

3.2 Instruments and Optical Elements

Figure 3.2 shows the setup for measuring H.

Spontaneous Parametric Down-Conversion

The pump laser[5] outputs light which is reflected off two mirrors and passes through a half-wave plate and quartz plate before reaching the down-conversion crystal.

Spontaneous parametric down-conversion (henceforth referred to as SPDC) is a process by which a photon is converted into two photons. A schematic of the process is shown in Figure 3.3. SPDC is a nonlinear optical process,[6] and occurs with a very low probability ($\sim 10^{-8}$).[7] Although this is not an efficient process, it is much more

[5] Power Technology, Inc. Laser Diode Control Unit Model #LDCU12/6931. Output wavelength: 405nm. Max power: 50mW. Safety goggles must be worn at all times when the laser is on.

[6] The polarization of a dielectric medium in the presence of an electric field is classically given by

$$P = \epsilon_0 \chi^{(1)} E + \epsilon_0 \chi^{(2)} E^2 + \epsilon_0 \chi^{(3)} E^3 + ...$$

For small electric field amplitudes, the linear ($\chi^{(1)}$) term dominates. However, when we have a large amplitude field (i.e., a laser beam), we must take into consideration the second-order polarization (the $\chi^{(2)}$ term). Even with large field amplitudes, however, nonlinear processed are still dominated by linear processes, and have a low probability of occurring.

[7] Suryawan [2009]

easily implemented and efficient than the atomic cascade used by Aspect et al. [1982].

SPDC occurs when an electron is excited by an incident photon, but as the electron returns to the ground state, it emits two photons rather than one.[8] We refer to the two output beams as the *signal* and *idler* beams.[9] Conservation of energy and momentum respectively require that

$$\omega_p = \omega_i + \omega_s \qquad \text{and} \qquad \mathbf{k}_p = \mathbf{k}_i + \mathbf{k}_s.$$

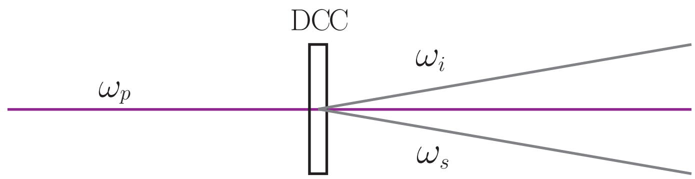

Figure 3.3: Spontaneous Parametric Down-Conversion. DCC denoted the down-conversion crystal.

We employ *degenerate* down-conversion, which is a down-conversion process by which $\omega_i = \omega_s$. Since we pump the crystal with a 405 nm laser, the two photons are output with a wavelength of 810 nm.[10]

We achieve SPDC through the use of two stacked β-Barium Borate (BBO) crystals.[11] Each are 0.5mm thick, and they are mounted so that one is rotated 90° relative to the other about the axis perpendicular to the large face. This allows for downconversion of both horizontally and vertically polarized photons:

$$a\,|H\rangle + b\,e^{i\phi}\,|V\rangle \xrightarrow{SPDC} a\,|V\rangle_1\,|V\rangle_2 + b\,e^{i\phi}\,|H\rangle_1\,|H\rangle_2$$

The signal and idler beams are deflected by an angle of 3°.[12] The half-wave plate preceding the BBO crystal served to adjust the relative amplitudes of a and b; the quartz plate adjusts the phase ϕ. We adjust the phase so that $\phi = 0$, i.e. so that the light is exactly linearly polarized.

[8] SPDC can be thought of as the reverse of *sum-frequency mixing.* For two electric fields, with frequencies ω_1 and ω_2, the second-order polarization is

$$\begin{aligned} P^{(2)} &= \epsilon_0 \chi^{(2)} E_1 \cos(\omega_1 t)\, E_2 \cos(\omega_2 t) \\ &= \epsilon_0 \chi^{(2)} E_1\, E_2 \left(\cos(\omega_1 - \omega_2)t + \cos(\omega_1 + \omega_2)t \right) \end{aligned}$$

And so if we observe the second-order field, we see waves with frequency $\omega_1 + \omega_2$. In SPDC, the opposite occurs; we have a wave of frequency $\omega_1 + \omega_2$, and observe it separate into two beams of frequency ω_1 and ω_2.

[9] It is arbitrary which beam is labeled signal and which is labeled idler.

[10] Not all photons have a wavelength of precisely 810 nm. Rather, the signal and idler photons have a spread which is centered around 810 nm. Note that if $\lambda_i > 810$ nm, then $\lambda_s < 810$ nm, in order to conserve energy.

[11] Purchased from Photop Technologies.

[12] See Suryawan [2009] or Vitullo [2007] for a derivation.

Rochon Polarizers

The signal and idler beams leave the BBO crystal and pass through a series of optical elements which measure their polarization in a given axis. We measure the polarization of photons by using a Rochon polarizer.[13] Rochon polarizers separate a homogeneously polarized field by transmitting and deflecting perpendicular components of the incident light (see Figure 3.4).[14] Rochon prisms are available which deflect at a range of angles (from $< 1°$ to $> 15°$). We use a prism with 15° deflection.[15] We place half-wave plates directly before the Rochon prims, which allows us to effectively modify the polarization axis of the prism, and thus define along which axis we measure.

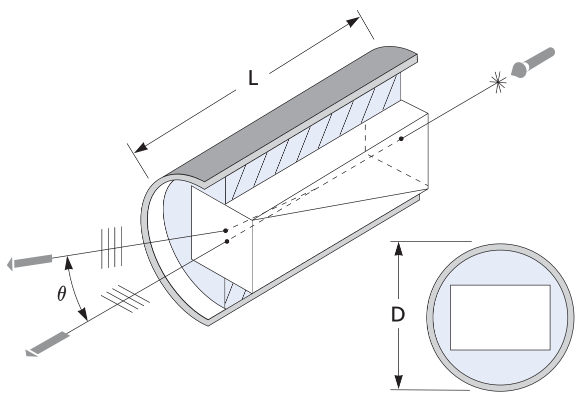

Figure 3.4: A Rochon Polarizer. Used with permission from CVI Melles Griot.

The light then is coupled into fiber by collimators A, B, A′, and B′. The polarizers are set up so that A and B correspond to horizontal polarization in the given axis, and A′ and B′ correspond to vertical polarization.[16]

Single Photon Detection and Coincidence Counting

After the light is coupled into fibers by the collimators, it is sent to a single-photon counting module, or SPCM.[17] In the SPCMs, an avalanche photodiode converts single-photon signals into a 5V TTL (transistor-transistor logic) electric pulse.[18] These detectors have a rise time of ~ 20 ns.

[13] Rochon prisms are preferable to the standard beam-splitter cubes because they are more efficient.
[14] Which components are transmitted and which are reflected depend upon the orientation of the polarizer.
[15] CVI Melles Griot part number RCHP-15.0-CA- 670-1064.
[16] We refer to "detectors" A, B, A′, and B′, but this is misleading, as the detection actually occurs in the SPCMs. When we say we align the detectors, we really mean that we align the collimators which then send the coupled light to the SPCM.
[17] Perkin Elmer SPCM-AQ4C.
[18] An excellent explanation of this process is given in Suryawan [2009].

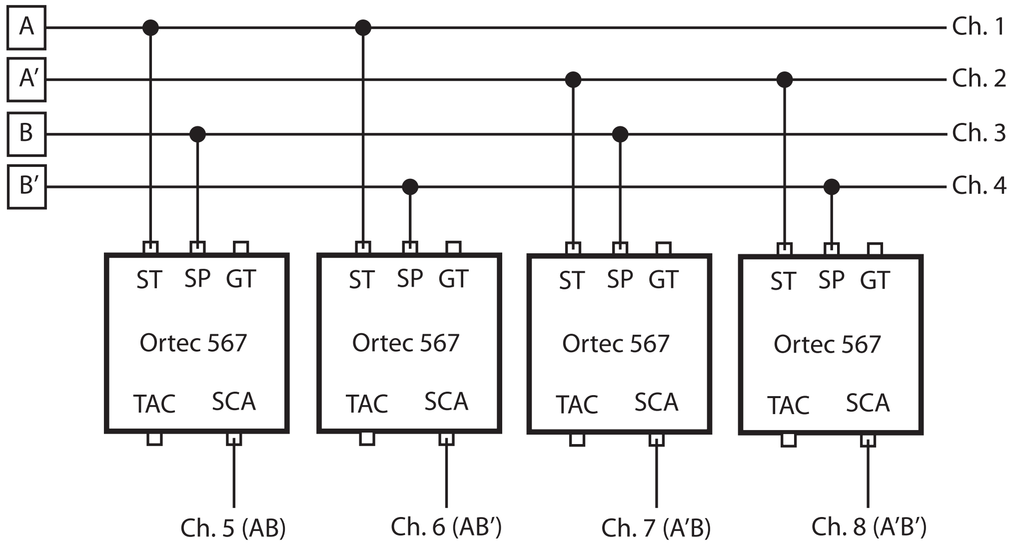

Figure 3.5: Coincidence electronics. Ch. 1-4 are raw counts from detectors; Ch. 5-8 are coincidence counts.

The TTL signals from the SPCMs are sent to the a time-to-amplitude converter/single-channel analyzer.[19] Figure 3.5 shows how the detectors are connected to the four TAC/SCA units to give the four coincidence counts AB, AB′, A′B, and A′B′.

Each TAC/SCA has three inputs: start (ST), stop (SP), and gate (GT). When the start gate is triggered by a TTL pulse, the SCA waits for a user-defined amount of time (called the window). If it receives a pulse on the stop input within this window, then a coincidence event is registered, and the SCA outputs a TTL pulse. The TAC operates similarly, but will output a signal whose amplitude is proportional to the time between the start and stop inputs. The SCA can also be operated in a mode where start pulses are only accepted when there is a simultaneous pulse on the gate input. Since we only detect coincidence between two inputs, the use of the gate input is unnecessary.

The A and A′ detectors are connected to the coincidence electronics via a 3 ft BNC cable. The electric signals travel at $\sim 2/3\,c$ through the cable, so the signal takes $\sim$ 4.5 ns to travel through the cable. The B and B′ detectors are connected via a 13 ft BNC cable, and we measure a 14.7 ns delay between the outputs of the two cables. The A and A′ detectors are then connected to the start inputs, and the B and B′ detectors are connected to the stop inputs. Although the SAC output is ultimately what we measure, observing the TAC output can be useful is setting the SCA window.[20]

[19]Ortec 567 TAC/SCA

[20]Our SAC window for this experiment was $\sim$ 2.4 ns.

Monitoring Coincidence via LabVIEW

The outputs of the detectors and the SCAs are sent to a counting board,[21] which interfaces with LabVIEW. For his quantum mechanics laboratory,[22] Mark Beck has written LabVIEW VIs which were used not only for the Hardy experiment, but also for the two preliminary experiments.[23]

In order to use Mark's program in our laboratory, however, we had to modify it to interface with the equipment at hand, which was slightly different from Mark's equipment. The issue was the controller which controls the motorized waveplates; whereas Mark calls for the use of a GPIB controller, we had a serial controller[24] on hand. The sub-VIs which send or receive commands from the controller had to be rewritten in order to work with the NSC-SB.[25]

With these modifications, the VI was capable of either actively monitoring the four coincidence counts and the corresponding probability[26] with a variable update time, or taking data in a fully automated fashion, moving the waveplates and calculating the appropriate probabilities automatically.[27]

3.3 Alignment of Optical Elements

As with most optical experiments, the most challenging aspect of Hardy's test is proper alignment of the optical elements. I will provide a thoroughly detailed description of my alignment procedure in the hopes that it may be helpful to future students who undertake to perform Hardy's test.

An essential tool in the alignment of the system is the use of an alignment laser.[28] The output of this laser is fiber-coupled backwards through the detectors. This allows us to "aim" the detectors and ensure that they are roughly facing the downconversion crystal. The alignment laser also allows us to ensure that the surfaces of the half-wave plate and Rochon crystals are normal to the direction of propagation, through observation of the reflection of the beam off of these optical elements (this procedure is described below in detail).

[21] National Instruments PCI-6602.

[22] Whitman College course no. P385

[23] These are the Grangier Experiment and Single-Photon Interference; see the Appendices for details.

[24] Newport NSC-SB

[25] It is evident why Mark no longer uses this controller - it was perpetually problematic, frequently misinterpreting commands sent, or ignoring them altogether.

[26] Mark's VI is set up to measure P_{AB}, which is given by

$$P_{AB} = \frac{N_{AB}}{N_{AB} + N_{AB'} + N_{A'B} + N_{A'B'}}$$

[27] Although it was possible to use the VI in this manner, it was ultimately prohibitively error-prone; see Ch. 5 for details.

[28] Thorlabs LPS-785-FC 785nm laser diode powered by ILX Lightwave LDX-3412 precision current source.

Initial Alignment

First, we must set up the pump and down-conversion section of the apparatus by mounting the laser, half-wave plate, quartz plate, and down-conversion crystal on the optical table. The quartz plate is mounted on a rotation stage. It is desirable to leave space between the optical elements so that irises can be inserted for alignment purposes.

We must ensure that the half-wave plate is normal to the direction of propagation to the laser beam. To do this, insert an iris in the beam preceding the half wave plate. Tighten the iris until the beam is small. Most of the laser light that passes through the wave plate will be transmitted, but a small portion will be reflected; we can observe the reflection of the wave plate and tell its angle relative to the propagation of the beam. When the reflection lines up exactly with where the beam passes through the iris, we know that the wave-plate is at normal incidence. Insert the quartz plate, and repeat this to ensure that the quartz plate is also at normal incidence.[29]

Using the same technique described above, check that the beam strikes the down-conversion crystal is at normal incidence. Interestingly, the crystal does not down-convert at perfectly normal incidence - it prefers to be tilted from the normal at an angle of approximately 3°.[30] For now, however, we will put the crystal at normal incidence - it will later be detuned from the normal to maximize the flux of down-converted photons. Again, most of the light that is incident upon the crystal will be transmitted; only a small portion of the light is down-converted.[31]

Place a beam-stop approximately 1.5 m past the down-conversion crystal.[32] Detectors A and B are placed at a 3° deflection from the transmitted beam, roughly 2 m from the down-conversion crystal.[33] Although not absolutely necessary, it is incredibly helpful to place one of the detectors on a translation stage, so that minor adjustments in horizontal position can be made without disturbing the alignment of the detector. Using the alignment laser, the detectors are aimed so that the alignment laser passes through the down-conversion crystal. Note that this does not ensure that we will detect the output of the crystal - it only ensures that *if* the crystal outputs photons in the direction of the detector, they will be seen by the detector.

[29] We will eventually rotate the quartz plate to adjust phase; for alignment, however, it is best to leave it at normal incidence.

[30] That is, each crystal prefers to be tilted 3° in the direction of the accepted pump polarization.

[31] I will often refer to the "transmitted beam;" by this I mean the beam of light that is transmitted by the crystal.

[32] In this discussion, I will order the optical elements by the direction of propagation of the laser beam - thus, the "first" wave plate is the one above the downconversion crystal in Figure 3.2, the beam stop is "after" downconversion crystal, etc.

[33] The distance from the downconversion crystal to the detectors does not directly affect our measurements; however, due to the small deflection of the down-converted photons, the length from the crystal to the detectors must be large enough that the appropriate half-wave plates and polarizers can be fit into both arms of the apparatus.

Detecting Entanglement

We are now ready to find entangled photon pairs. With the detectors and coincidence electronics properly connected to the computer, turn on the laser and start up the appropriate Labview VI.[34] By careful adjustment of the detectors, maximize the counts of both detectors. The long-pass filters must also be aligned, which can be done roughly via the alignment laser, with fine-tuning done by monitoring detection counts in LabVIEW. What we are interested in is not maximizing the raw counts on each detector, however - we wish to maximize the coincidences on the detectors, which correspond to entangled photon pairs.

When one is not observing photon pairs, it can be difficult to say precisely which factor is to blame. There are three main issues that arise in observing entanglement:

1. The detectors are not placed correctly. That is, the detectors are not placed symmetrically at 3° deflection from the path of the beam transmitted by the down-conversion crystal. This is done roughly by simple geometry and measurement - the fine alignment is done via a translation stage.

2. The down-conversion crystal is not at the proper angle. Each of the two stacked crystals are very sensitive to tilt in the direction parallel to their output polarization. For example, if one has low $|V\rangle\,|V\rangle$ outputs, then the vertical tilt of the crystal should be carefully adjusted. Once the detectors are at approximately the right location, the crystal should be tilted until some photon pairs are observed. We can then maximize the number of pairs observed by translating one of the detectors, and then re-tuning the crystal.

3. The pump beam is not at polarized properly. If we have adjusted only one axis of the down-conversion crystal (i.e. we have only tuned one of the two stacked crystals properly), then the pump beam must be appropriately polarized to pump the crystal and obtain photon pairs. The output of the laser used in this experiment is polarized; rotation of the half-wave plate preceding the crystal will modify this polarization. Use this to maximize the number of photon pairs observed.

4. The detectors are not at the right height. Ensure that both detectors are at the same height, and that the beam enters the crystal and hits the beam stop at roughly this height. Fine adjustments to the height of the beam can be made on the mirrors which reflect the pump beam. Coincidence detection is *extremely* height sensitive, so be careful in adjusting the mirrors.

I recommend addressing the four issues given in the above order, but it will be necessary to fine-tune the system by iterating these adjustments (i.e. place the detectors, tune the crystal, rotate the pump beam, adjust the height of the beam, re-adjust

[34] Most of Beck's Labview VIs given on his website are capable of counting two-detector coincidence. At this stage of alignment, I recommend using a simpler VI than the one designed to measure H, as this VI carries a lot of unnecessary baggage.

detectors, re-tune crystal, etc). When the system is properly aligned, one should be able to observe ∼400 coincidences per second.[35]

Measuring Polarization of Photon Pairs

With this stage of alignment completed, we are ready to insert our Rochon polarizers and align our A′ and B′ detectors. First, insert the Rochon polarizers approximately 1 m from the down-conversion crystal, in the path of the down-converted photons. Using the alignment laser, ensure that the detectors A and B are still aimed at the crystal so that the output of the alignment laser is passing through the crystal. Since the output of the alignment laser is unpolarized, the alignment beam should split as it exits the Rochon polarizer, with one output continuing straight and one deflecting by 15°. Now, couple the alignment laser into detector A′. By placing this detector in *front* of the Rochon polarizer (so that the output of the alignment laser propagates in the same direction as the output of the down-conversion crystal) we can check in which direction the polarizer is deflecting. It is easiest to have the polarizers deflecting outward, away from the path of the transmitted beam.

Now place the detector approximately where it should detect photons deflected by 15° from detector A. Of course, we must make sure that the primed detectors are at the same height as the unprimed detectors. The alignment laser coming out of detector A′ should be aimed so that it enters the Rochon polarizer. We are now concerned with the *deflected* beam, rather than the transmitted. We wish the deflected beam (from detector A′) to follow precisely the same path as the transmitted beam (from detector A). We do this by observing the beam in two locations. We insert an iris between the crystal and the Rochon polarizer, as close as possible to the polarizer. We will observe the alignment beam here and as it passes through the down-conversion crystal.

Couple the alignment beam into detector A. We have ensured that this detector is well aligned in the previous section, so we will use the path of this alignment beam as a reference. Tighten the iris, and place it so that the alignment beam passes through the iris. Now, couple the alignment laser into detector A′. Aim the detector so that the portion of the beam deflected by the Rochon polarizer passes through the iris. If the beam passes through the iris *and* passes through the crystal at the same point as our reference beam, we are done. However, it most likely will pass through the iris and then not pass through the crystal. This means we must translate our detector horizontally in order to match the two paths. Move the detector slightly, and then again aim it at through the polarizer at the iris. Repeat this until the alignment beam passes through both the iris and the crystal.

When the paths of the alignment beams are matched, the A′ detector is roughly aligned. Repeat this procedure with detector B′.[36] We are now ready to turn on the

[35]This number will vary with the power of the laser used and the efficiency of the down-conversion crystal. The quoted number is for the specific equipment used in this laboratory.

[36]Of course, it doesn't matter which primed detector you align first.

laser and start the Hardy VI to test our alignment.[37]

At this point, insert the half-wave plates that sit in front of the polarizers. By rotating these wave-plates, we should be able to pass the coincidence counts entirely back and forth between the detectors. If the photon pairs are not being observed on the detectors, recheck the rough alignment issues (height, aiming of alignment beam, horizontal placement of detector) and then use the adjustment knobs to fine-tune the detector. Now, we should be able to pass the coincidence counts between the detectors perfectly (i.e. start out with 400 AB coincidence/sec, then by adjusting both half-wave plates 45°, have 400 A′B′ coincidence/sec).

A final consideration is determining the "experimental zeros" of the half-wave plates. Set the half-wave plates to their zero hash. Since the down-conversion crystal only outputs $|H\rangle|H\rangle$ or $|V\rangle|V\rangle$ pairs, we should measure zero AB′ and A′B coincidence. If there is a non-zero measurement for these coincidences, make slight adjustments to the wave plates until the setting is found where these coincidences are minimized. This is the "experimental zero" of the waveplate. This is very important to take into consideration when setting the angles of measurement during a data run.

Tuning the State

In order to maximize our Hardy violation, we use the state

$$|\Psi\rangle = 0.421\ |H\rangle_1 |H\rangle_2 + 0.907\ |V\rangle_1 |V\rangle_2 . \tag{3.9}$$

Since the probability of measuring $|H\rangle_1 |H\rangle_2$ coincidence is $|a|^2 \approx 0.2$ (and the probability of measuring $|V\rangle_1 |V\rangle_2$ coincidence is $|b|^2 \approx 0.8$), we have the state above when the ratio of $|H\rangle_1 |H\rangle_2$ to $|V\rangle_1 |V\rangle_2$ coincidence is $1 : 4$.

With both wave plates set to 0°, tune the state so that the ratio of AB ($|H\rangle|H\rangle$) to A′B′ ($|V\rangle|V\rangle$) coincidence is 1:4.[38] We measure

$$H = P(\beta,-\beta) - P(-\alpha,\alpha) - P(\beta,\alpha^{\perp}) - P(-\alpha^{\perp},-\beta) \tag{3.10}$$

where $\alpha \approx 34°$ and $\beta \approx 18°$.

The second step of our alignment is adjusting the phase of the state. Set the wave plates so that $P(-\alpha,\alpha)$ is being measured;[39] we then adjust the angle of the quartz plate to minimize this probability. We can now iterate through each probability we wish to measure, minimizing (for $P(-\alpha,\alpha)$, $P(\beta,\alpha^{\perp})$, and $P(-\alpha^{\perp},-\beta)$) or maximizing (for $P(\beta,-\beta)$) appropriately. This can be done by slight adjustments of measurement angle,[40] or my adjusting the quartz plate or half-wave plate that affect the pump beam. We are now ready to take data.

[37] At this point, we must use the Hardy VI, as it can measure the four coincidences we are interested, unlike the other VIs which measure two or three coincidences.

[38] Of course, one can run the experiment in a symmetric system with the ratios 4:1 - I found the 1:4 state easier to work with due to asymmetry in my alignment

[39] That is to say, adjust the hash mark on the half wave plates to $-\alpha/2$ and $\alpha/2$, respectively.

[40] Note that the measurement angles are not independent. For example if we change α to minimize $P(-\alpha,\alpha)$, we must measure $P(\beta,\alpha^{\perp})$ with respect to the new α as well.

Chapter 4

Results

Data was taken using a 10 s integration time. 10 data points were taken for each probability. Using the angles $\alpha = 34°$ and $\beta = 18°$, we found

$$\begin{aligned} \mathrm{P}(\beta, -\beta) &= 0.158 \pm 0.003 \\ \mathrm{P}(-\alpha, \alpha) &= 0.019 \pm 0.001 \\ \mathrm{P}(\beta, \alpha_\perp) &= 0.030 \pm 0.003 \\ \mathrm{P}(-\alpha_\perp, -\beta) &= 0.046 \pm 0.002 \end{aligned}$$

We calculated H by summing the probabilities (with appropriate negative signs) element-by-element through the lists. This gives us ten values of H. We then divide the standard deviation by $\sqrt{n}$, where n is the number of data points in the distribution, to find the uncertainty:

$$\text{Uncertainty}(H) = \frac{\sigma_H}{\sqrt{10}}. \tag{4.1}$$

This gives

$$H = 0.063 \pm 0.001, \tag{4.2}$$

violating the predictions of local realism by 63 uncertainties.[1]

4.1 Discussion

While progressing through the experiment, I noted a few areas in which the apparatus for this experiment could have been improved to yield better data.

Alignment

In aligning the system prior to taking measurements, I encountered some difficulties. Prior to my entry into the lab, there were three collimator-detectors on the table. I

[1] We have violated local realism by 15 standard deviations, which corresponds to 63 uncertainties by Equation (4.1).

built a fourth (the A′ detector), which involved assembling the collimator and coupling it into the detector through a long-pass filter. After the filter was aligned, it was found to be $\sim 25\%$ more efficient than the other three detectors.

More generally, the alignment was of varying quality on each detector. Translation stages on all four detectors would have been very helpful, because it would have allowed for small horizontal adjustments without disturbing the overall alignment of the system. Without these, however, I was unable to fine-tune my horizontal position (key to detecting entanglement) without disturbing the overall alignment of the system.

Newstep Motor Controller

Although I did modify the LabVIEW VI provided by Beck so that it would work with the NewStep motor controller that we had, it was never fully functional. The data taken was done manually, adjusting wave plates by hand and performing statistical analysis in *Mathematica*. There were many issues with the controller, of which I will list here the most notable.

1. Most problematically, the motors would often simply move to positions other than those given in the commands sent to it. This seemed to be an issue with the provided LabVIEW drivers.

2. Channel 2 of the NSC-SB was often problematic, and would not initialize or shut down properly, leading to errors to the second waveplate.

3. When homing, the rotators would undergo a seemingly random number of rotations before settling in the home position. This would sometimes last for a dozen or more full rotations.

In personal correspondence, Beck insisted that none of these issues arose when one used the GPIB switchbox; however, it was prohibitively expensive and, obviously, not necessary to purchase one for this experiment. However, the manual data acquisition prohibited the use of long integration times.[2]

Even with these issues, a clear violation of local realism is seen, a testament to the simple and robust experimental design of Carlson et al. [2006].

[2] In Carlson et al. [2006], a 45 s integration time is used, which contributes greatly to the incredibly low standard deviation of the data. When using a 20 s integration time, they find a 32 standard deviation violation.

Chapter 5

Conclusion

We have shown a clear empirical violation of local realism. Performing such an experiment in a modest undergraduate laboratory was significantly more difficult and expensive until Mark Beck put forth the ingenious experimental designs that allow this and other quantum mechanics experiments to be performed with relative ease and low cost.

5.1 Future Experiments

We have not yet exhausted the possibilities of implementing Mark's designs as Reed theses. There are two experiments which would follow up on the apparatus and techniques gained from Hardy's test. I will now briefly describe these experiments.

5.1.1 Quantum Eraser with Polarization-Entangled Photons

This experiment is based on a subtle modification of the apparatus used in the single-photon interference experiment. Beginning with the setup used in the Hardy experiment, a polarization interferometer is inserted in the signal arm. Since the down-conversion crystal is producing entangled photon pairs, measurement of the polarization in one arm tells us about the polarization in the other arm. Using this fact, we can erase "which-path" information for the signal arm by measuring the polarization of the idler arm. The setup for this experiment is shown in Figure 5.1. This experiment could be well implemented as a thesis.

5.1.2 Bell Inequalities Test using Polarization Entangled Photons

This experiment is based on the work of Dehlinger and Mitchell [2002], and uses the same apparatus as that used in Hardy's test. The quantity S is measured,[1] and local realism is violated for $S \geq 2$. This experiment could easily be performed using the Hardy apparatus; unfortunately, time constraints prevented me from doing so. This

[1] See Clauser et al. [1970].

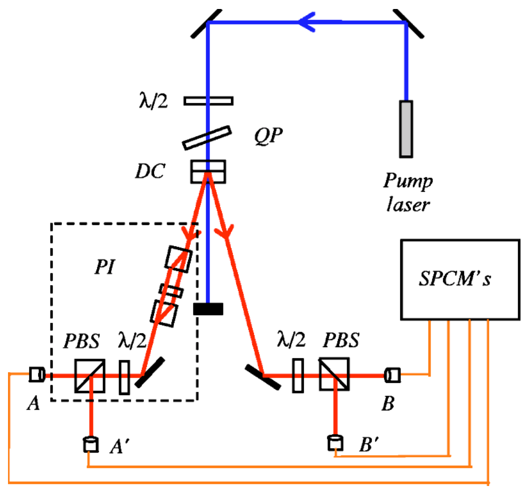

Figure 5.1: Experimental apparatus for the Quantum Eraser experiment. Taken from Gogo et al. [2005]

could even be implemented in the Advanced Laboratory course for juniors; with the setup from this thesis present, there is little new alignment required.

Performing these experiments in an undergraduate laboratory is truly an invaluable learning experience. Shedding light[2] on the mysterious behavior of quantum particles through experimentation brings the message home to students that these are not simply numbers on a chalkboard; they are empirically verifiable physical realities. Also, performing these experiments is an incredible learning experience in optical laboratory techniques for the undergraduate experimenter. It is my sincere hope that more experiments like those presented here are undertaken by future Reed students of physics.

[2]Via Power Technology, Inc. Laser Diode Control Unit Model #LDCU12/6931. Output wavelength: 405nm. Max power: 50mW. Safety goggles must be worn at all times when the laser is on.

Appendix A

Preliminary Experiments

During the first semester of this thesis project, I performed two preliminary experiments, which helped develop the laboratory techniques necessary to successfully implement Hardy's Test. There are many subtleties to the experimental techniques used here, and skillful application of these techniques is the difference between inconclusive and significant data. These experiments were originally performed as thesis projects by Vitullo [2007] and Suryawan [2009]. I will present here an extremely brief description of the two experiments; for a more thorough treatment, the reader is referred to the respective theses (available in the Reed College Library).[1]

A.1 The Grangier Experiment

Experiment

In this experiment, we aim to verify the particle nature of light. In specific, we show that a photon incident on a beamsplitter will be either transmitted or reflected, but not both. Figure A.1 shows the experimental setup via which we measure this. We use three detectors; a gate (G) detector in the idler beam, and transmitted (T) and reflected (R) detectors in the signal beam. We only consider coincidences events between two or more detectors; we measure GT, GR, and GTR coincidence. This allows us to be sure that our measurements are made on photons from the signal beam and not stray photons from the room or dark counts of the detectors. We define a quantity called the *second-order coherence* as

$$g^{(2)}(0) = \frac{P_{GTR}}{P_{GT}\,P_{GR}}.^{2} \tag{A.1}$$

Classical mechanics demands that $g^{(2)}(0) \geq 1$; however, there are certain quantum

[1]For the Grangier experiment, the reader may also see Thorn et al. [2004].

[2]The probabilities are calculated by

$$P_{GT} = \frac{N_{GT}}{N_G},$$

and similarly for P_{GR} and P_{GTR}.

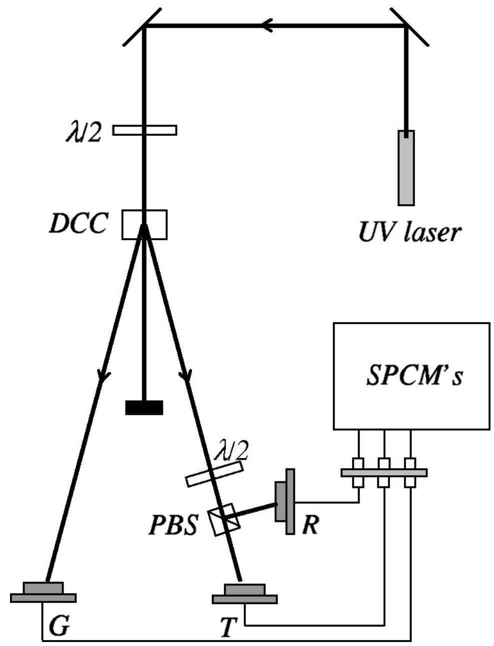

Figure A.1: Diagram of the setup for the Grangier experiment, taken from Thorn et al. [2004]. Note that we used a non-UV laser which operated at 405 nm.

states for which it is predicted that $g^{(2)}(0) < 1$. For the single photon state, quantum mechanics predicts that $g^{(2)}(0) = 0$.[3]

This experiment familiarizes one with the detector and BBO crystal alignment. A single BBO crystal is used to produce photon pairs. In order to acquire data, one must also become familiar with the operation of coincidence electronics, including setting the SCA window and considering TAC vs. SCA operation modes. One also learns to use the alignment laser (not shown in Figure A.1) to align the collimators.

Results

We found $g^{(2)}(0) = 0.063 \pm 0.008$, which violates the classical inequality $g^{(2)}(0) \geq 1$ by over 110 standard deviations.

[3]Thorn et al. [2004]

A.2 Single-Photon Interference

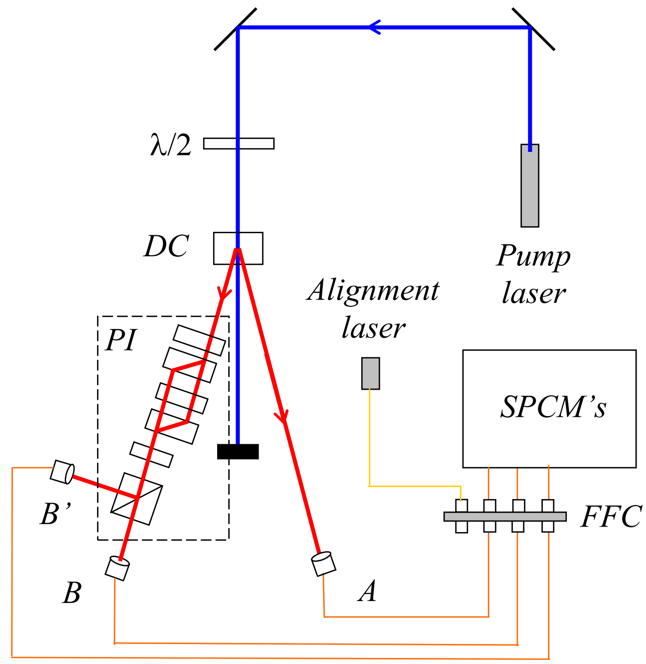

Figure A.2: Diagram of the setup for Single-Photon Interference. Taken from Beck [2008]

Experiment

In this experiment, a polarization interferometer is added in the signal arm of the Grangier setup. Using the half-wave plates in the interferometer, we can create a state in which it is impossible to know which path the photon takes through the interferometer, i.e. both paths have a 50% probability. In this state, we say we have erased "which-path" information. As we make minute changes in the length of one arm of the interferometer, we expect to see a sinusoidal variation in the gated number of counts N_{AB} and $N_{AB'}$ due to interference between the two arms of the interferometer. When we have full "which-path" information, we expect to see no interference. This verifies that a single photon will pass through both arms of the interferometer, so long as we do not measure which arm it passes through. We also observe the second-order coherence $g^{(2)}(0)$ to ensure that we remain as close as possible to the ideal single-photon state.

This experiment gives further experience in alignment of optical elements. Initial alignment of the interferometer requires that all optical elements be normal to the direction of beam propagation, which is ensured through careful observation of the reflection of the alignment laser off of the incident surfaces of the various optical elements.[4] The adjustment of the length of one arm is done in LabVIEW via a motor controller and a linear actuator connected to a kinetic mount.[5] The experimenter is thus familiarized with the techniques necessary to successfully implement this interface to achieve computer-controlled motors.[6]

Results

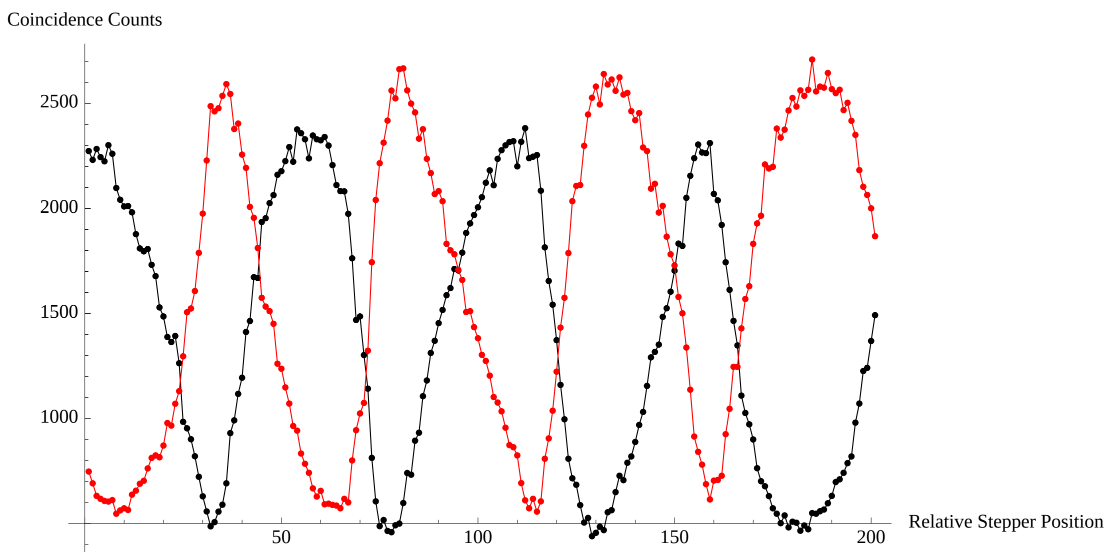

Figure A.3: Results demonstrating single-photon interference. The light data is $N_{AB'}$ and the dark data is N_{AB}.

Figure A.3 shows the interference when the "which-path" is not available. The visibility of the interference was $V_{AB} = 0.70$ and $V_{AB'} = 0.66$.[7] The second-order coherence of this data is $g^{(2)}(0) = 0.05 \pm 0.07$. Although not shown, we observed no interference[8] when "which path" information was available.

---

[4]This technique is described in greater detail in §3.3.

[5]Motor controller: Newport NSC-SB. Actuator: Newport NSA-12. Kinetic mount: Newport SN100-F2KN

[6]This basically amounts to a lot of rebooting of the controller, as well as using the software bundled with the NSC-SB rather than LabVIEW itself to "home" the motors (i.e. return to the zero setting).

[7]The *visibility* of an interference pattern I is defined as

$$V_I = \frac{\max(I) - \min(I)}{\max(I)}$$

There is no interference when $V = 0$; perfect interference occurs when $V = 1$.

[8]Variations in counts were due to random noise; if one wished to calculate the visibility in this case, it would be $\sim 5\%$.

Appendix B

Derivation of Bell's Inequality

We will now present a brief derivation of Bell's Inequality (see Bell [1964]). Although not directly applicable to the experiment at hand, Bell's paper was the inspiration for myriad tests of local realism, and is cited in virtually all papers on the topic.[1] This section follows the exposition given in Griffiths [2005].

Suppose we observe the decay of a π^0 particle into an positron and electron:

$$\pi^0 \longrightarrow e^+ + e^- \tag{B.1}$$

Since the net spin of the particle is zero before the decay, we know that the spin of the positron will be perfectly anticorrelated with the spin of the electron, when they are measured in the same basis. For simplicity, we measure the spin in units of $\hbar/2$: we thus measure $+1$ for spin up and -1 for spin down. Now suppose we measure the spin of the positron in the basis defined by the unit vector $\mathbf{a}$, and the spin of the electron in the basis defined by the unit vector $\mathbf{b}$. We can then take the average *product* of these spin measurements, which we will denote $P(\mathbf{a}, \mathbf{b})$, the average product of spin measurements along $\mathbf{a}$ and $\mathbf{b}$, for e^+ and e^- respectively. In this notation, our statement of anticorrelation of spin becomes:

$$P(\mathbf{a}, \mathbf{a}) = -1 \tag{B.2}$$

Equivalently, if measured along the opposite axes, the spins are perfectly correlated:

$$P(\mathbf{a}, -\mathbf{a}) = 1. \tag{B.3}$$

Quantum mechanics predicts that, generally,[2]

$$P(\mathbf{a}, \mathbf{b}) = -\,\mathbf{a} \cdot \mathbf{b} \tag{B.4}$$

This result will be shown to be incompatible with any hidden variable theory. For a hidden variable theory, the measurement of one particle does not affect the result

[1] Including, of course, Hardy [1993]

[2] This argument is taken from Wheeler [2009]. We choose our axes so that $\mathbf{a}$ lies along the z-axis and $\mathbf{b}$ is in the x-z plane. Then $S_a^{(1)} = S_z^{(1)}$ and $S_b^{(2)} = \cos\theta S_z^{(1)} + \sin\theta S_x^{(2)}$. We are to calculate

of a measurement on the other; that is to say, the result of both measurements are completely determined by the hidden variable (call it λ). We can thus construct two functions of λ, one for the positron and one for the electron, which give us spin measurement results independently:

$$A(\mathbf{a}, \lambda) = \pm 1 \qquad\qquad B(\mathbf{b}, \lambda) = \pm 1 \tag{B.5}$$

Since our spins are anticorrelated, we know that

$$A(\mathbf{a}, \lambda) = -B(\mathbf{a}, \lambda) \tag{B.6}$$

Hang on to your hats; we now dive into some mathematical manipulation in order to arrive at a relation between the spin products P for three different combinations of directions. If $\rho(\lambda)$ is the normalized probability distribution of λ, then the average spin measurement $P(\mathbf{a}, \mathbf{b})$ is given by

$$P(\mathbf{a}, \mathbf{b}) = \int \rho(\lambda) \left[A(\mathbf{a}, \lambda) B(\mathbf{b}, \lambda)\right] d\lambda \tag{B.7}$$

In light of Equation (B.6), we can rewrite this all in terms of A:

$$P(\mathbf{a}, \mathbf{b}) = -\int \rho(\lambda) \left[A(\mathbf{a}, \lambda) A(\mathbf{b}, \lambda)\right] d\lambda \tag{B.8}$$

Suppose that $\mathbf{c}$ is some third unit vector. Then

$$P(\mathbf{a}, \mathbf{b}) - P(\mathbf{a}, \mathbf{c}) = -\int \rho(\lambda) \left[A(\mathbf{a}, \lambda) A(\mathbf{b}, \lambda) - A(\mathbf{a}, \lambda) A(\mathbf{c}, \lambda)\right] d\lambda \tag{B.9}$$

Given that $A(\mathbf{b}, \lambda)^2 = 1$, we can write

$$P(\mathbf{a}, \mathbf{b}) - P(\mathbf{a}, \mathbf{c}) = -\int \rho(\lambda) \left[1 - A(\mathbf{b}, \lambda) A(\mathbf{c}, \lambda)\right] A(\mathbf{a}, \lambda) A(\mathbf{b}, \lambda) d\lambda. \tag{B.10}$$

$\langle 00| S_a^{(1)} S_b^{(2)} |00\rangle$. Neglecting factors of $\hbar/2$,

$$\begin{aligned}
S_a^{(1)} S_b^{(2)} |00\rangle &= \frac{1}{\sqrt{2}} \left\{ S_z^{(1)} (\cos\theta S_z^{(2)} + \sin\theta S_x^{(2)})(\uparrow\downarrow - \downarrow\uparrow) \right\} \\
&= \frac{1}{\sqrt{2}} \{ (S_z \uparrow)(\cos\theta S_z \downarrow + \sin\theta S_x \downarrow) - (S_z \downarrow)(\cos\theta S_z \uparrow + \sin\theta S_x \uparrow) \} \\
&= \frac{1}{\sqrt{2}} \{ \uparrow (\cos\theta(-\downarrow) + \sin\theta \uparrow - (-\downarrow (\cos\theta \uparrow + \sin\theta \downarrow) \} \\
&= \cos\theta \frac{1}{\sqrt{2}} (-\uparrow\downarrow + \downarrow\uparrow) + \sin\theta \frac{1}{\sqrt{2}} (\uparrow\uparrow + \downarrow\downarrow) \\
&= -\cos\theta |00\rangle + \sin\theta (|11\rangle + |1\ -1\rangle)
\end{aligned}$$

So

$$\begin{aligned}
\langle S_a^{(1)} S_b^{(2)} \rangle = \langle 00| S_a^{(1)} S_b^{(2)} |00\rangle &= \langle 00| \{ -\cos\theta |00\rangle + \sin\theta (|11\rangle + |1\ -1\rangle) \} \\
&= -\cos\theta \langle 00|00\rangle \qquad \text{by orthogonality} \\
&= -\cos\theta
\end{aligned}$$

$\square$

We now must note two facts about Equation (B.10). First, the product $A(\mathbf{a},\lambda)A(\mathbf{b},\lambda)$ is always between $+1$ and -1, which is to say

$$|A(\mathbf{a},\lambda)A(\mathbf{b},\lambda)| \leq 1. \tag{B.11}$$

Also, in light of the above, we know that

$$\rho(\lambda)\left[1 - A(\mathbf{b},\lambda)A(\mathbf{c},\lambda)\right] \geq 0, \tag{B.12}$$

which puts us in a position to write

$$|P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})| \leq \int \rho(\lambda)\left[1 - A(\mathbf{b},\lambda)A(\mathbf{c},\lambda)\right] d\lambda, \tag{B.13}$$

which can be put more simply as

$$|P(\mathbf{a},\mathbf{b}) - P(\mathbf{a},\mathbf{c})| \leq 1 + P(\mathbf{b},\mathbf{c}). \tag{B.14}$$

Equation (B.14) is the famous inequality put forth by Bell in 1964. It is easily violated by the quantum mechanical prediction of Equation (B.4). Suppose that we define our three vectors in the usual cartesian form:

$$\mathbf{a} = \hat{x} \tag{B.15}$$

$$\mathbf{b} = \hat{y} \tag{B.16}$$

$$\mathbf{c} = \frac{1}{\sqrt{2}}\left(\hat{x} + \hat{y}\right). \tag{B.17}$$

In this case, quantum mechanics says that

$$P(\mathbf{a},\mathbf{b}) = 0, \qquad P(\mathbf{a},\mathbf{c}) = P(\mathbf{b},\mathbf{c}) = -0.707, \tag{B.18}$$

which give us our contradiction,

$$0.707 \nleq 1 - 0.707 = 0.293 \tag{B.19}$$

The implications of Bell's Theorem were not fully understood at first, but it would ultimately open the door to empirical tests of local realism.

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